A PROBLEM ON PARTITIONS WITH A PRIME MODULUS $p \geqq 3$ (1)

BY

PETER HAGIS, JR.

I. Introduction

1. Let $p \geqq 3$ be a prime, and let $a = \{a_1, a_2, \cdots, a_r\}$, where $1 \leqq a_i \leqq (p-1)/2$, be a set of r distinct integers. We are concerned with the problem of determining $p_a(n)$, the number of partitions of a positive integer n into summands congruent to $\pm a_i \pmod p$. Using the circle dissection method of Hardy [2](2) and Rademacher [7; 8], Lehner [3] succeeded in obtaining a convergent series for $p_a(n)$ in the special case where $p=5$ and $r=1$. This result was generalized by Livingood [4] who found convergent series for $p_a(n)$ for any prime $p>3$ and $r=1$. Both Lehner and Livingood also obtained asymptotic formulas for $p_a(n)$. Petersson [5; 6] has recently obtained many results concerning $p_a(n)$ and other much more general partition functions. His method differs radically from those of the above mentioned authors and makes use of algebraic and group theoretic properties of the generating functions rather than analytic processes. Grosswald [1] has considered a generalization of the present problem where the set a is replaced by a set of m distinct least positive residues modulo p, and $p_a(n)$ is taken as the number of partitions of n into summands congruent modulo p to elements of this set. The present problem is the special case which Grosswald designates as symmetrical. Due to the asymmetry in the general case, only asymptotic formulas are obtained. The method employed is a variation of the circle dissection process.

In the present paper the procedure of Rademacher and Lehner is utilized and a convergent series for $p_a(n)$ is obtained. The effect of adjoining the element p to the set a, so that summands congruent to $0 \pmod p$ are admissible, is then studied, and Rademacher's formula for $p(n)$, the number of unrestricted partitions of n, is later derived as a direct consequence. Asymptotic formulas are also developed and some special cases considered. While many of the results obtained are not new, it is felt that the present investigation is not without value. Theorems first proven by the function-theoretic method of Petersson have been shown to be obtainable by the simpler and more gen-

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(1) This paper, in expanded form, was presented to the faculty of the University of Pennsylvania in June, 1959 in partial fulfillment of the requirements for the degree of Doctor of Philosophy. The author wishes to express his gratitude to Professor Emil Grosswald not only for suggesting the problem but also for giving most generously of his time during its solution.

(2) Numbers enclosed in square brackets refer to the bibliography.

eral circle integration method. We have here a clear indication that partition functions of great generality can be studied by using relatively simple analytic procedures.

II. The Transformation Equations

2. We consider the generating function

$$F_a(x) = \prod_{i=1}^{r}\left(\prod_{m=0}^{\infty}(1 - x^{pm+a_i})^{-1}\prod_{m=1}^{\infty}(1 - x^{pm-a_i})^{-1}\right) = 1 + \sum_{n=1}^{\infty} p_a(n)x^n \tag{2.1}$$

which is convergent in the interior of the unit circle. In what follows it will be necessary to determine the behavior of $F_a(x)$ in the neighborhood of a rational point on a circle concentric to the unit circle and of radius less than 1. Therefore, we take

$$x = \exp\{2\pi ih/k - 2\pi z/k\} \tag{2.2}$$

where

$$\Re(z) > 0, \qquad (h, k) = 1, \qquad 0 \leqq h < k.$$

Our first objective is to derive a transformation equation for $F_a(x)$. Two cases must be considered. If $p \mid k$ then we study the transformation $x \to x'$, where

$$x' = \exp\{2\pi ih'/k - 2\pi z^{-1}/k\} \tag{2.3}$$

and h' is a fixed solution of

$$hh' \equiv -1 \pmod{k}. \tag{2.4}$$

If $p \nmid k$ we study the transformation $x \to x''$, where

$$x'' = \exp\{2\pi iH'/k - 2\pi z^{-1}/K\} \tag{2.5}$$

and

$$HH' \equiv -1 \pmod{k}, \qquad K = pk, \qquad H = ph. \tag{2.6}$$

Now, $F_a(x)$ is regular and different from 0 in the unit circle. Therefore, $\log F_a(x)$ is single-valued in the same region if we select the branch given by $\log F_a(0) = 0$ and we may define

$$\begin{aligned} G_a(x) = \log F_a(x) &= -\sum_{i=1}^{r}\left(\sum_{m=0}^{\infty}\log(1 - x^{pm+a_i}) + \sum_{m=1}^{\infty}\log(1 - x^{pm-a_i})\right) \\ &= \sum_{i=1}^{r}\left(\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{x^{(pm+a_i)n}}{n} + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{x^{(pm-a_i)n}}{n}\right). \end{aligned} \tag{2.7}$$

3. We first consider the case $p\mid k$. For $i=1, 2, \cdots, r$ set

$$(3.1)\qquad \begin{gathered} pm \pm a_i = qk + \mu_i, \qquad 0 < \mu_i < k, \qquad \mu_i \equiv \pm a_i \pmod p, \\ n = tk + \nu, \qquad \nu = 1, 2, \cdots, k, \\ q,\ t = 0, 1, 2, \cdots \end{gathered}$$

in (2.7). We then have

$$(3.2)\qquad \begin{aligned} G_a(x) = &\sum_{i=1}^{r}\sum_{\mu_i}\sum_{\nu=1}^{k} \exp\{2\pi i h\mu_i\nu/k\} \\ &\cdot \sum_{q,t} (tk+\nu)^{-1}\exp\{-2\pi z(qk+\mu_i)(tk+\nu)/k\}.\text{(3)} \end{aligned}$$

Applying Mellin's formula we obtain

$$(3.3)\qquad \begin{aligned} G_a(x) = &\frac{1}{2\pi i k}\sum_{i=1}^{r}\sum_{\mu_i,\nu} \exp\{2\pi i h\mu_i\nu/k\} \\ &\cdot \int_{(3/2)} \frac{\Gamma(s)}{(2\pi z k)^s}\zeta(s, \mu_i/k)\zeta(s+1, \nu/k)ds \end{aligned}$$

where $\zeta(s, w)$ is the Hurwitz zeta function and (α) indicates an integration from $\alpha - i\infty$ to $\alpha + i\infty$. Using the transformation equation for $\zeta(s, w)$ and the identity $\Gamma(s)\Gamma(1-s) = \pi/\sin \pi s$ we have finally

$$(3.4)\qquad \begin{aligned} G_a(x) = &\frac{1}{4\pi i k^2}\sum_{i=1}^{r}\sum_{\mu_i,\nu,\lambda} \cos\frac{2\pi h\mu_i\nu}{k}\cos\frac{2\pi\lambda\mu_i}{k}\int_{(3/2)}\frac{\zeta(1-s, \lambda/k)\zeta(1+s, \nu/k)ds}{z^s\cos(\pi s/2)} \\ &+ \frac{1}{4\pi k^2}\sum_{i=1}^{r}\sum_{\mu_i,\nu,\lambda} \sin\frac{2\pi h\mu_i\nu}{k}\sin\frac{2\pi\lambda\mu_i}{k}\int_{(3/2)}\frac{\zeta(1-s, \lambda/k)\zeta(1+s, \nu/k)ds}{z^s\sin(\pi s/2)} \end{aligned}$$

where $\lambda = 1, 2, \cdots, k$.

We now introduce the summation letters $\tilde{\mu}_i$ for $i = 1, 2, \cdots, r$ by requiring that

$$(3.5)\qquad \mu_i \equiv h'\tilde{\mu}_i \pmod k, \qquad 0 < \tilde{\mu}_i < k$$

where h' is given by (2.4). Multiplying this congruence by h

$$(3.6)\qquad \tilde{\mu}_i \equiv -h\mu_i \pmod k.$$

Since $p\mid k$ and since $\mu_i \equiv \pm a_i \pmod p$ we have

$$(3.7)\qquad \tilde{\mu}_i \equiv \pm ha_i \equiv \pm b_i \pmod p$$

where

$$(3.8)\qquad b_i \equiv ha_i \pmod p \quad \text{or} \quad b_i \equiv -ha_i \pmod p,$$

(3) $i = (-1)^{1/2}$ except when used as a subscript.

whichever yields $0<b_i \leqq (p-1)/2$.

We now define the set b,

$$b = \{b_1, b_2, \cdots, b_r\} \tag{3.9}$$

noting that $b_i \neq b_j$ if $i \neq j$ since this equality would imply that $a_i \equiv \pm a_j \pmod{p}$ since $(h, p)=1$.

Changing s to $-s$ and μ_i to $\tilde{\mu}_i$ in (3.4) yields

$$\begin{aligned} G_a(x) &= \frac{1}{4\pi i k^2} \sum_{i=1}^{r} \sum_{\tilde{\mu}_i, \nu, \lambda} \cos \frac{2\pi \tilde{\mu}_i \nu}{k} \cos \frac{2\pi h' \lambda \tilde{\mu}_i}{k} \int_{(-3/2)} \frac{\zeta(1+s, \lambda/k)\zeta(1-s, \nu/k)ds}{z^{-s} \cos(\pi s/2)} \\ &+ \frac{1}{4\pi k^2} \sum_{i=1}^{r} \sum_{\tilde{\mu}_i, \nu, \lambda} \sin \frac{2\pi \tilde{\mu}_i \nu}{k} \sin \frac{2\pi h' \lambda \tilde{\mu}_i}{k} \int_{(-3/2)} \frac{\zeta(1+s, \lambda/k)\zeta(1-s, \nu/k)ds}{z^{-s} \sin(\pi s/2)} \\ &= \int_{(-3/2)} I_1 ds + \int_{(-3/2)} I_2 ds. \end{aligned} \tag{3.10}$$

If the path of integration in (3.10) is changed to $\Re(s)=3/2$ the integrals remain convergent and we find that the sums in (3.10) then take the same form as those in (3.4) except that $\tilde{\mu}_i$ has replaced μ_i, h' has replaced h, z^{-1} has replaced z, and λ and ν have been interchanged. Since both integrals in (3.10) converge to zero if the path of integration is taken between $3/2+it$ and $-3/2+it$ and $|t| \to \infty$, we have, applying Cauchy's Residue Theorem, and recalling (2.3) and (3.9)

$$G_a(x) = G_b(x') - 2\pi i(R_1 + R_2). \tag{3.11}$$

Here $R_1 = \sum \text{Res } I_1$, $R_2 = \sum \text{Res } I_2$ where $-3/2 < \Re(s) < 3/2$.

The procedure involved in calculating R_1 and R_2 is similar to that employed by Lehner and Livingood and all details are therefore omitted. We obtain finally

$$R_1 = \frac{1}{12pki}(Az - B/z), \tag{3.12}$$

where

$$A = \sum_{i=1}^{r} (p^2 - 6a_i p + 6a_i^2) \tag{3.13}$$

and

$$B = \sum_{i=1}^{r} (p^2 - 6b_i p + 6b_i^2); \tag{3.14}$$

$$R_2 = -\frac{1}{2}\sigma_a(h, k) \tag{3.15}$$

where

$$\sigma_a(h, k) = \sum_{i=1}^{r} \sum_{\mu_i} ((\mu_i/k))((h\mu_i/k)). \tag{3.16}$$

Here,

$$((x)) = x - [x] - \frac{1}{2} + \frac{1}{2}\delta(x) \tag{3.17}$$

where $\delta(x) = 1$ if x is an integer and 0 otherwise.

From (3.11), (3.12), (3.15) and exponentiation we have the transformation equation for $F_a(x)$ for the case $p \mid k$.

$$F_a(x) = \omega_a(h, k) \exp\left\{\frac{\pi}{6pk}(B/z - Az)\right\} F_b(x') \tag{3.18}$$

where

$$\omega_a(h, k) = \exp\{\pi i \sigma_a(h, k)\}. \tag{3.19}$$

4. We now turn our attention to the case $p \nmid k$. As before x is given by (2.2) while x'', K, H, and H' are defined by (2.5) and (2.6). Referring back to (2.7), for $i = 1, 2, \cdots, r$ we set

$$\begin{gathered} pm \pm a_i = qK + \mu_i, \qquad 0 < \mu_i < K, \qquad \mu_i \equiv \pm a_i \pmod{p}, \\ n = tk + \nu, \qquad \nu = 1, 2, \cdots, k, \\ q, t = 0, 1, 2, \cdots. \end{gathered} \tag{4.1}$$

Following exactly the same procedure as in §3 we find that

$$\begin{aligned} G_a(x) = {} & \frac{1}{4\pi i k K} \sum_{i=1}^{r} \sum_{\mu_i, \nu, \lambda} \cos\frac{2\pi h \mu_i \nu}{k} \cos\frac{2\pi \lambda \mu_i}{K} \int_{(3/2)} \frac{\zeta(1-s, \lambda/K)\zeta(1+s, \nu/k)ds}{z^s \cos(\pi s/2)} \\ & + \frac{1}{4\pi k K} \sum_{i=1}^{r} \sum_{\mu_i, \nu, \lambda} \sin\frac{2\pi h \mu_i \nu}{k} \sin\frac{2\pi \lambda \mu_i}{K} \int_{(3/2)} \frac{\zeta(1-s, \lambda/K)\zeta(1+s, \nu/k)ds}{z^s \sin(\pi s/2)} \end{aligned} \tag{4.2}$$

where $\lambda = 1, 2, \cdots, K$.

We now define μ_i^* for $i = 1, 2, \cdots, r$ by

$$\mu_i \equiv pH'\mu_i^* \pmod{k}, \qquad 0 < \mu_i^* \leqq k. \tag{4.3}$$

Since $ph = H$ and since $HH' \equiv -1 \pmod{k}$ we have

$$\mu_i^* \equiv -h\mu_i \pmod{k}. \tag{4.4}$$

Introducing μ_i^* and changing s to $-s$ in (4.2) yields

$$G_a(x) = \frac{1}{4\pi ikK}\sum_{i=1}^{r}\sum_{\mu_i,\nu,\lambda}\cos\frac{2\pi\mu_i^*\nu}{k}\cos\frac{2\pi\lambda\mu_i}{K}\int_{(-3/2)}\frac{\zeta(1+s,\lambda/K)\zeta(1-s,\nu/k)ds}{z^{-s}\cos(\pi s/2)}$$

$$(4.5)\quad +\frac{1}{4\pi kK}\sum_{i=1}^{r}\sum_{\mu_i,\nu,\lambda}\sin\frac{2\pi\mu_i^*\nu}{k}\sin\frac{2\pi\lambda\mu_i}{K}\int_{(-3/2)}\frac{\zeta(1+s,\lambda/K)\zeta(1-s,\nu/k)ds}{z^{-s}\sin(\pi s/2)}$$

$$= \int_{(-3/2)} I_1 ds + \int_{(-3/2)} I_2 ds.$$

If we shift the path of integration in (4.5) to $\Re(s)=3/2$ both integrals remain convergent and we have by Cauchy's Residue Theorem

$$(4.6)\qquad G_a(x) = J_a(x'') - 2\pi i(R_1 + R_2).$$

Here $J_a(x'')$ is the right-hand member of (4.5) with $(-3/2)$ replaced by $(3/2)$, and R_1 and R_2 denote the residues at the poles with $-3/2<\Re(s)<3/2$ for the integrands I_1 and I_2 respectively.

Our immediate objective is to verify that the notation $J_a(x'')$ is justified. If we reverse the steps involved in obtaining (4.5) from (2.7) we obtain

$$(4.7)\qquad \begin{aligned} J_a(x'') = \sum_{i=1}^{r}\sum_{\mu_i,\lambda}\exp\{2\pi i\lambda\mu_i/K\}\sum_{q,t=0}^{\infty}\frac{1}{qK+\lambda} \\ \cdot\exp\{-2\pi(tk+\mu_i^*)(qK+\lambda)/Kz\}. \end{aligned}$$

By (4.1) μ_i runs through $K/p=k$ values which are $\equiv a_i \pmod p$. No two of these values can be congruent (mod k). For if $\mu_i-\mu_i' = jp>0$ and also $\mu_i-\mu_i' = j'k$ it would follow, since $p\nmid k$, that $\mu_i-\mu_i' = mpk=mK$. But this implies that $\mu_i > K$ which is impossible. Similarly μ_i runs through k values $\equiv -a_i \pmod p$, no two of which are congruent (mod k). We conclude that μ_i runs through a complete residue system (mod k) twice. Therefore, there is for each $i=1, 2, \cdots, r$ an α_i such that

$$(4.8)\qquad \alpha_i k \equiv a_i \pmod p, \qquad 0<\alpha_i<p.$$

In conjunction with (4.3) this yields

$$(4.9)\qquad \mu_i \equiv pH'\mu_i^* \pm \alpha_i k \pmod K$$

where the $\pm$ agrees with $\mu_i \equiv \pm a_i \pmod p$.

Applying this result in (4.7)

$$(4.10)\qquad \begin{aligned} &\exp\{2\pi i\lambda\mu_i/K\} \\ &= \exp\{2\pi i(\pm\lambda\alpha_i/p + \lambda H'\mu_i^*/k)\} \\ &= \exp\{\pm 2\pi i\alpha_i(qK+\lambda)/p\}\exp\{2\pi iH'(tk+\mu_i^*)(qK+\lambda)/k\}. \end{aligned}$$

By (4.3) and the remarks following (4.7), μ_i^* runs through the values $1, 2, \cdots, k$ twice in some order. Therefore, setting

$$qK + \lambda = m,$$
$$(4.11)\qquad tk + \mu_i^* = n,$$
$$\rho_i = \exp(2\pi i\alpha_i/p), \qquad \bar{\rho}_i = \exp(-2\pi i\alpha_i/p)$$

in (4.7) and using (4.10) we have

$$(4.12)\qquad \begin{aligned} J_a(x'') &= \sum_{i=1}^{r}\left(\sum_{m,n=1}^{\infty}\frac{\rho_i^m x''^{mn}}{m} + \sum_{m,n=1}^{\infty}\frac{\bar{\rho}_i^m x''^{mn}}{m}\right) \\ &= \log H_a(x'') \end{aligned}$$

where

$$(4.13)\qquad \begin{aligned} H_a(x) &= \prod_{i=1}^{r}\left(\prod_{n=1}^{\infty}(1-\rho_i x^n)^{-1}\prod_{n=1}^{\infty}(1-\bar{\rho}_i x^n)^{-1}\right) \\ &= 1 + \sum_{\nu=1}^{\infty} c_a^{(k)}(\nu)x^{\nu}. \end{aligned}$$

We remark that since $|\rho_i| = 1$ we have, by a comparison with $\prod_{n=1}^{\infty}(1-x^n)^{-1}$, the generating function of the Euler partition function, the convergence of this series in the interior of the unit circle. Indeed, the convergence is uniform in k.

Turning to the calculation of R_1 and R_2 in (4.6) we find that

$$(4.14)\qquad R_1 = \frac{Az}{12pki} - \frac{r}{12pkzi} + \frac{1}{2\pi i}\sum_{i=1}^{r}\log(2\sin\pi\alpha_i/p),$$

and

$$(4.15)\qquad R_2 = -\frac{1}{2}t_a(h, k)$$

where

$$(4.16)\qquad t_a(h, k) = \sum_{i=1}^{r}\sum_{\mu_i}((\mu_i/K))((h\mu_i/k)).$$

From (4.6), (4.12), (4.14), (4.15) and exponentiation we obtain the transformation equation for $F_a(x)$ for the case $p\nmid k$. Writing

$$(4.17)\qquad \chi_a(h, k) = \exp\{\pi i t_a(h, k)\}$$

we combine this result with (3.18) to obtain

THEOREM 1. *$F_a(x)$ satisfies the transformation equation*

$$(4.18)\qquad F_a(\exp\{2\pi ih/k - 2\pi z/k\}) = \omega_a(h, k) \exp\left\{\frac{\pi}{6pk}(B/z - Az)\right\} \cdot F_b(\exp\{2\pi ih'/k - 2\pi/kz\})$$

if $p|k$, and the equation

$$(4.19)\qquad F_a(\exp\{2\pi ih/k - 2\pi z/k\}) = \frac{1}{2^r}\chi_a(h, k) \prod_{i=1}^{r} \csc \pi\alpha_i/p \cdot \exp\left\{\frac{\pi}{6pk}(r/z - Az)\right\} H_a(\exp\{2\pi iH'/k - 2\pi/Kz\})$$

if $p\nmid k$.

III. Estimates of Two Exponential Sums

5. In the sequel it will be necessary to have some information concerning the magnitude of certain exponential sums involving $\omega_a(h, k)$ and $\chi_a(h, k)$ and taken over a reduced system of residues modulo k. The trivial estimate $O(k)$ will not suffice for our purposes so that we must now undertake a study of these sums. Our procedure will be to reduce them to Kloosterman sums. The method used is essentially that of Lehner, and therefore most of the details will be omitted. Considering first the case $p|k$ we find that

$$(5.1)\qquad 6pk\sigma_a(h, k) = \sum_{i=1}^{r}\Big(2h\{2k^2 + 3k(2a_i - p) + A_i\} - 3k\{k - 2p + 2a_i + 2c_i\} - 12p\sum_{M_i}\mu_i[h\mu_i/k]\Big).$$

Here, M_i $(i=1, 2, \cdots, r)$ is the set consisting of those μ_i such that $\mu_i \equiv +a_i \pmod p$ and $A_i = p^2 - 6a_ip + 6a_i^2$. c_i is defined by

$$(5.2)\qquad c_i = \begin{cases} b_i & \text{if } \{ha_i, p\} \leqq (p-1)/2, \\ p - b_i & \text{if } \{ha_i, p\} > (p-1)/2, \end{cases}$$

where $\{s, t\} \equiv s \pmod t$, $0 < \{s, t\} \leqq t$.

From (5.1) we see that $6pk\sigma_a(h, k)$ is always an integer. We shall determine some congruences satisfied by it. Assume temporarily that $p > 3$. Then it follows that

$$(5.3)\qquad 6pk\sigma_a(h, k) \equiv 0 \pmod 3 \qquad \text{if } 3\nmid k.$$

$$(5.4)\qquad 6pk\sigma_a(h, k) \equiv \sum_{i=1}^{r}(2p + 2a_i + 2c_i - 3) \pmod 4 \qquad \text{if } 2\nmid k.$$

We also have

$$(5.5)\qquad \begin{aligned} 6hkp\sigma_a(h, k) = h^2 \sum_{i=1}^{r} \{2k^2 + 3k(2a_i - p) + A_i\} + (B - rk^2) \\ - 3hk \sum_{i=1}^{r} (2c_i - p) - 12kp \sum_{i=1}^{r} S_i \end{aligned}$$

where $S_i = (1/2) \sum_{M_i} [h\mu_i/k]([h\mu_i/k]+1)$ is an integer.

Now let $12p = fG$ where f is the greatest divisor of $12p$ prime to k. There are four possible values for f (recall that $p > 3$).

$$(5.6)\qquad \begin{aligned} &(k, 12p) = p, && f = 12,\ G = p, \\ &\left.\begin{aligned}(k, 12p) &= 2p \\ (k, 12p) &= 4p\end{aligned}\right\}, && f = 3,\ \ G = 4p, \\ &(k, 12p) = 3p, && f = 4,\ \ G = 3p, \\ &\left.\begin{aligned}(k, 12p) &= 6p \\ (k, 12p) &= 12p\end{aligned}\right\}, && f = 1,\ \ G = 12p. \end{aligned}$$

We take h' now so that $hh' \equiv -1 \pmod{Gk}$. This is possible since all primes in G divide k, so that $(h, k) = 1$ implies that $(h, Gk) = 1$. We then multiply (5.5) by $-h'$, noting that $Gk \mid 2k^2$, to obtain

$$(5.7)\qquad 6pk\sigma_a(h, k) \equiv hu - h'v - 3k \sum_{i=1}^{r} (2c_i - p) \pmod{Gk}$$

where

$$(5.8)\qquad \begin{aligned} u &= \sum_{i=1}^{r} \{3k(2a_i - p) + A_i\}, \\ v &= B - rk^2. \end{aligned}$$

Using (5.3) and (5.4) we readily verify that

$$(5.9)\qquad 6pk\sigma_a(h, k) \equiv \sum_{i=1}^{r} (6a_i + 6c_i + 6p - 3) \pmod{f}.$$

If $p = 3$ the above procedure must be modified slightly. In this case $r = 1$, $a_1 = a = 1$, $c_1 = c$, $A_1 = A = -3$. By (5.4), which holds for $p = 3$, we have $6pk\sigma_a(h, k) \equiv (2c - 3) \pmod{4}$ if $2 \nmid k$. Setting $12p = 36 = fG$ as before we now have only two possible values for f. If $(k, 36) = 3, 9$ then $f = 4$, $G = 9$. If $(k, 36) = 6, 12, 18, 36$ then $f = 1$, $G = 36$. Now (5.7) holds with $u = 2k^2 - 3k - 3$. (5.9) is now $6pk\sigma_a(h, k) \equiv (21 + 6c) \pmod{f}$. This congruence is obvious if $f = 1$. If $f = 4$, then k is odd and we have $6pk\sigma_a(h, k) \equiv (2c - 3) \equiv (21 + 6c) \pmod{4}$. Therefore, (5.9) is true for all $p \geqq 3$.

We now return to the general case $p \geqq 3$. Define the integers ϕ and Γ by setting

$$f\phi \equiv 1 \pmod{Gk}, \quad Gk\Gamma \equiv 1 \pmod{f}. \tag{5.10}$$

Then, from (5.7), (5.9), (5.10) we have

$$6pk\sigma_a(h, k) \equiv f\phi\left(uh - vh' - 3k\sum_{i=1}^{r}(2c_i - p)\right) + Gk\Gamma\left(\sum_{i=1}^{r}(6a_i + 6c_i + 6p - 3)\right) \pmod{12pk}. \tag{5.11}$$

By (3.19), (5.11), and setting

$$A^* = \sum_{i=1}^{r} a_i, \quad C^* = \sum_{i=1}^{r} c_i \tag{5.12}$$

we have

$$\begin{aligned}\omega_a(h, k) &= \exp\left\{\frac{2\pi i}{12pk}\,6pk\sigma_a(h, k)\right\} \\ &= \exp\left\{2\pi i\left(\frac{\Gamma}{f}(6A^* + 6C^* + 6rp - 3r) - \frac{3\phi}{G}(2C^* - rp) + \frac{\phi}{Gk}(uh - vh')\right)\right\}.\end{aligned} \tag{5.13}$$

6. Turning to the case $p \nmid k$ we find that

$$6pkt_a(h, k) = \sum_{i=1}^{r}\left(2h\{2K^2 + 3K(2a_i - p) + A_i\} - 3k\{K - p + 2a_i - 2\alpha_i\} - 12\sum_{M_i}\mu_i[h\mu_i/k]\right) \tag{6.1}$$

which shows that $6pkt_c(h, k)$ is always an integer. Assuming temporarily that $p > 3$ we find that

$$6pkt_a(h, k) \equiv 0 \pmod{3} \qquad \text{if } 3 \nmid k, \tag{6.2}$$

$$6pkt_a(h, k) \equiv \sum_{i=1}^{r}(p - pk + 2a_i + 2\alpha_i) \pmod{4} \qquad \text{if } 2 \nmid k, \tag{6.3}$$

$$6pkt_a(h, k) \equiv 0 \pmod{p}, \tag{6.4}$$

and

$$
\begin{aligned}
6hkpt_a(h, k) &= h^2 \sum_{i=1}^{r} \{2K^2 + 3K(2a_i - p) + A_i\} + r(1 - k^2) \\
&\quad + 6hk \sum_{i=1}^{r} \alpha_i - 12k \sum_{i=1}^{r} T_i
\end{aligned} \tag{6.5}
$$

where $T_i = (1/2) \sum_{M_i} [h\mu_i/k]([h\mu_i/k]+1)$ is an integer.

Now let $12p = Fg$ where F is the greatest divisor of $12p$ prime to k. There are four possible values for F (recall that $p > 3$).

$$
\begin{array}{lll}
(k, 12p) = 1, & F = 12p, & g = 1, \\
\left.\begin{array}{l}(k, 12p) = 2 \\ (k, 12p) = 4\end{array}\right\}, & F = 3p, & g = 4, \\
(k, 12p) = 3, & F = 4p, & g = 3, \\
\left.\begin{array}{l}(k, 12p) = 6 \\ (k, 12p) = 12\end{array}\right\}, & F = p, & g = 12.
\end{array} \tag{6.6}
$$

We take h' now so that $hh' \equiv -1 \pmod{gk}$. This is possible since all primes in g divide k. Noting that $gk \mid 2k^2$ and multiplying (6.5) by $-h'$ we obtain

$$
6pkt_a(h, k) \equiv hu - h'v + 6k \sum_{i=1}^{r} \alpha_i \pmod{gk} \tag{6.7}
$$

where

$$
\begin{aligned}
u &= \sum_{i=1}^{r} \{3K(2a_i - p) + A_i\}, \\
v &= r(1 - k^2).
\end{aligned} \tag{6.8}
$$

Using (6.2), (6.3), (6.4) we readily verify that

$$
6pkt_a(h, k) \equiv 9p^2 \sum_{i=1}^{r} (2a_i + 2\alpha_i + p - pk) \pmod{F}. \tag{6.9}
$$

If $p = 3$ the above procedure must be modified slightly. In this case $r = 1$, $a_1 = a = 1$, $\alpha_1 = \alpha$, $A_1 = A = -3$, $\mu_1 = \mu$, $M_1 = M$. By (6.3), which holds for $p = 3$, we have $6pkt_a(h, k) \equiv (5 - 3k + 2\alpha) \pmod{4}$ if $2 \nmid k$. For $p = 3$ we easily verify (see (5.2) in [4]) that

$$
\begin{aligned}
2kt_a(h, k) &= \frac{2}{9} h\{2K^2 - 3K - 3\} - \frac{4}{3} \sum_{M} \mu[h\mu/k] - 2k\left\{\frac{K}{6} - \frac{1}{2} + \frac{1}{3}\right\} \\
&\quad + \frac{2}{3} \alpha k.
\end{aligned}
$$

Since $\mu\equiv 1 \pmod 3$ we have $\sum_M \mu[h\mu/k]\equiv\sum_M [h\mu/k] \pmod 3$. Using (3.17) we have

$$\sum_M [h\mu/k] = -\sum_M ((h\mu/k)) + \sum_M h\mu/k - \frac{1}{2}\sum_M 1 + \frac{1}{2}\sum_M \delta(h\mu/k).$$

Since, by the discussion immediately following (4.7), the μ in M run through a complete residue system modulo k, we have by a theorem of Rademacher ((2.31) in [10]) $\sum_M ((h\mu/k))=0$. Therefore,

$$\begin{aligned}\sum_M [h\mu/k] &= \frac{h}{k}\sum_M \mu - k/2 + 1/2\\ &= \frac{h}{k}(K^2/6 - K/6) - k/2 + 1/2\\ &= 3hk/2 - h/2 - k/2 + 1/2.\end{aligned}$$

Recalling that $K=3k$ and that $\alpha k\equiv 1 \pmod 3$ we have

$$\begin{aligned}2kt_a(h, k) &= -2h/3 + 2h/3 + 2k/3 - 2/3 - 2k/3 + 2/3 + I\\ &= I\end{aligned}$$

where I is an integer. We conclude that

$$6pkt_a(h, k) \equiv 0 \pmod 9. \tag{6.10}$$

Setting $12p=36=Fg$ as before we now have only two possible values for F. If $(k, 36)=1$ then $F=36$, $g=1$. If $(k, 36)=2, 4$ then $F=9$, $g=4$. (6.7) holds as before. (6.9) is now $6pkt_a(h, k)\equiv 81(5-3k+2\alpha) \pmod F$. If $F=9$ this congruence is obvious. If $F=36$ the result follows from (6.3) and (6.10). Therefore, (6.9) is true for all $p\geqq 3$.

We now return to the general case $p\geqq 3$. Set

$$\begin{aligned}F\Phi &\equiv 1 \pmod{gk},\\ gk\gamma &\equiv 1 \pmod{F}.\end{aligned} \tag{6.11}$$

From (6.7), (6.9), (6.11) we have

$$\begin{aligned}6pkt_a(h, k) \equiv{}& F\Phi\left(uh - vh' + 6k\sum_{i=1}^{r}\alpha_i\right)\\ &+ 9gk\gamma p^2\sum_{i=1}^{r}(2a_i + 2\alpha_i + p - pk) \pmod{12pk}.\end{aligned} \tag{6.12}$$

By (4.17) and (6.12), and setting

$$\alpha^* = \sum_{i=1}^{r}\alpha_i \tag{6.13}$$

we have

$$\chi_a(h,k) = \exp\left\{\frac{2\pi i}{12pk}\, 6pkt_a(h,k)\right\}$$

$$(6.14)\qquad = \exp\left\{2\pi i\left(\frac{9p^2\gamma}{F}(2A^* + 2\alpha^* + rp - rpk) + \frac{6\Phi}{g}\alpha^* + \frac{\Phi}{gk}(uh - vh')\right)\right\}.$$

7. THEOREM 2. *The sum*

$$(7.1)\qquad A(n,\nu;k;d;\sigma_1,\sigma_2;a) = T = \sum_{h \bmod k}{}' \omega_a(h,k)\exp\{-2\pi i(hn - h'\nu)/k\},$$

where $h \equiv d \pmod{p}$, $p \nmid d$; $\sigma_1 \leqq h' < \sigma_2 \pmod{k}$, $0 \leqq \sigma_1 < \sigma_2 \leqq k$, $p \mid k$, *is subject to the estimate* $O(n^{1/3}k^{2/3+\epsilon})$ *uniformly in* ν, d, σ_1, σ_2, a.

Here σ_1, σ_2 are integers, h' is given by (2.4), and $\sum'$ indicates that h runs over integers prime to the modulus of the sum.

Proof. Taking G as given by (5.6) we have

$$T = \sum_{h \bmod k}{}' \omega_a(h,k)\exp\{-2\pi i(Gnh - G\nu h')/Gk\}.$$

Treating $\omega_a(h,k)$ as a function of h we see from (3.16) and (3.19) that it has period k. Therefore, if we change the modulus of the sum in T to Gk and always select h' so that $hh' \equiv -1 \pmod{Gk}$ the summands run through the same values but G times as often. Using (5.13) we then have

$$T = \frac{1}{G}\sum_{h \bmod Gk}{}' \exp\left\{2\pi i\left(\frac{\Gamma}{f}(6A^* + 6C^* + 6rp - 3r) - \frac{3\phi}{G}(2C^* - rp) + \frac{\phi}{Gk}(uh - vh')\right)\right\}\exp\{-2\pi i(Gnh - G\nu h')/Gk\}$$

where $0 \leqq h' < Gk$.

Since $h \equiv d \pmod{p}$ it follows from (5.2) that C^* remains constant for all permissible values of h in this expression for T. Therefore,

$$T = \frac{1}{G}\epsilon(a,k,d)\sum_{h \bmod Gk}{}' \exp\{2\pi i f(h)/Gk\}$$

where $|\epsilon(a,k,d)| = 1$ and $f(h) = (\phi u - Gn)h - (\phi v - G\nu)h'$.

Following Lehner's procedure we obtain

$$(7.2)\qquad T = O\Big(\log k \sum_{h \bmod Gk}{}' \cdot \exp\Big\{\frac{2\pi i}{Gk}((\phi u - Gn)h - (\phi v - Gv - Gl - skG/p)h')\Big\}\Big)$$

where the condition $h \equiv d \pmod p$ has been removed and s and l are integers.

The last sum in (7.2) is a complete Kloosterman sum and therefore, by a theorem of Salié (p. 264 in [11]), is subject to the estimate

$$O((\phi u - Gn, Gk)^{1/3}(Gk)^{2/3+\epsilon}).$$

Therefore,

$$(7.3)\qquad T = O((\phi u - Gn, Gk)^{1/3}(Gk)^{2/3+\epsilon} \log k).$$

Since $(f, Gk) = 1$, $(\phi u - Gn, Gk) = (f\phi u - fGn, Gk)$. But $fG = 12p$ and by (5.10) $f\phi u = u + mGk$. Therefore, we conclude that $(f\phi u - fGn, Gk) = (u - 12pn, Gk)$. Now, $(u - 12pn, Gk) \leqq (G(u - 12pn), Gk) = G(u - 12pn, k)$. By (5.8), (5.12), (3.13) $u = 3k(2A^* - rp) + A$ (if $p = 3$, $u = 2k^2 - 3k - 3$). We see immediately that $(u - 12pn, k) = (A - 12pn, k)$. Since $(A - 12pn, k) = O(n)$ we have by these remarks and (7.3)

$$T = O(n^{1/3}k^{2/3+\epsilon}).$$

THEOREM 3. *The sum*

$$(7.4)\qquad B(n, \nu; k; \sigma_1, \sigma_2; a) = U = \sum_{h \bmod k}{}' \chi_a(h, k) \exp\{-2\pi i(hn - H'\nu)/k\},$$

where $HH' \equiv -1 \pmod k$; $\sigma_1 \leqq h' < \sigma_2 \pmod k$, $0 \leqq \sigma_1 < \sigma_2 \leqq k$, $p \nmid k$, *is subject to the estimate* $O(n^{1/3}k^{2/3+\epsilon})$ *uniformly in* ν, σ_1, σ_2, a.

Proof. By (4.16) and (4.17) $\chi_a(h, k)$ has period k. Therefore, if we change the modulus of the sum to gk and always select H' so that $HH' \equiv -1 \pmod{gk}$ we have

$$(7.5)\qquad U = \frac{1}{g} \sum_{h \bmod gk}{}' \chi_a(h, k) \exp\{-2\pi i(gnh - g\nu H')/gk\}$$

where $0 \leqq h' < gk$.

If we now select h' so that $hh' \equiv -1 \pmod{gk}$ then $phh' \equiv -p \pmod{gk}$. Multiplying this congruence by H' and recalling that $ph = H$ we have $h' \equiv pH' \pmod{gk}$. Since $p \nmid gk$ there exists a δ such that $p\delta \equiv 1 \pmod{gk}$. Therefore,

$$(7.6)\qquad H' \equiv \delta h' \pmod{gk}.$$

By (7.5), (7.6), (6.14)

$$U = \frac{1}{g}\epsilon(a, k) \sum_{h \bmod gk}' \exp\{2\pi i f(h)/gk\}$$

where $|\epsilon(a, k)| = 1$ and $f(h) = (\Phi u - gn)h - (\Phi v - g\delta\nu)h'$.

Proceeding as in the proof of Theorem 2 we find that

$$U = O\left(\log k \sum_{h \bmod gk}' \exp\left\{\frac{2\pi i}{gk}((\Phi u - gn)h - (\Phi v - g\delta\nu - gl)h')\right\}\right).$$

This last sum is a complete Kloosterman sum and the rest follows exactly as before.

IV. A Convergent Series for $p_a(n)$

8. We now have the basic tools needed for an attack on our main problem, the determination of $p_a(n)$. The procedure followed in this section is that of Rademacher [8]. By Cauchy's integral formula and (2.1) we have

$$p_a(n) = \frac{1}{2\pi i}\int_C \frac{F_a(x)}{x^{n+1}}dx = \sum_{h,k;\, 0\le h<k\le N}' \frac{1}{2\pi i}\int_{\xi_{h,k}} \frac{F_a(x)}{x^{n+1}}dx$$

where $\xi_{h,k}$ are the Farey arcs of order N of C, the circle $|x| = \exp\{-2\pi N^{-2}\}$. $\sum'$ indicates that h runs through integers prime to k, with $h=0$ occurring only if $k=1$. If on the arc $\xi_{h,k}$ we introduce the variable ϕ by means of the equation $x = \exp\{-2\pi N^{-2} + 2\pi ih/k + 2\pi i\phi\}$ and write $w = N^{-2} - i\phi$, $z = wk$ we obtain

$$(8.1)\quad p_a(n) = \sum_{h,k;\, 0\le h<k\le N}' \exp\{-2\pi inh/k\} \int_{-\theta'_{h,k}}^{\theta''_{h,k}} F_a\left(\exp\left\{\frac{2\pi ih}{k} - \frac{2\pi z}{k}\right\}\right) \cdot \exp\{2\pi nw\}\,d\phi.$$

Here $\theta'_{h,k} = 1/k(k+k_1)$ and $\theta''_{h,k} = 1/k(k+k_2)$ where $h_1/k_1 < h/k < h_2/k_2$ are consecutive terms in the Farey series of order N.

We now split the sum over k in (8.1) into two parts, $p_a^{(1)}(n)$ and $p_a^{(2)}(n)$, according to whether $p|k$ or $p\nmid k$ respectively. We then have

$$(8.2)\qquad p_a(n) = p_a^{(1)}(n) + p_a^{(2)}(n).$$

Each of these sums will be treated separately. We consider first $p_a^{(1)}(n)$. Select β so that $a_1\beta \equiv 1 \pmod p$. By (3.8) we have $h \equiv b_1\beta \pmod p$ or $h \equiv -b_1\beta \pmod p$ both of which yield the same set $b = \{b_1, b_2, \cdots, b_r\}$. Therefore, as b_1 runs through its set of values for a given k we will pick up all the possible values of h by requiring that $h \equiv \pm b_1\beta \pmod p$, $0 \leqq h < k$, $(h, k) = 1$. Since, for a given a, b determines b_1 uniquely and, vice versa, b_1 determines b uniquely, we have from (8.1), (8.2), and (4.18) of Theorem 1

$$p_a^{(1)}(n) = \sum_{b_1=1}^{(p-1)/2} \sum_{h,k}{}' \omega_a(h, k) \exp\{-2\pi i n h/k\}$$
$$\cdot \int_{-\theta'}^{\theta''} \sum_{\nu=0}^{\infty} p_b(\nu) \exp\{2\pi i h' \nu/k\}$$
$$\cdot \exp\{-(\pi/k^2 w)(2\nu - B/6p) + \pi w(2n - A/6p)\} d\phi,$$

where $0 \leqq h < k \leqq N$, $h \equiv \pm b_1\beta (\bmod p)$, $k \equiv 0 \pmod p$.

For a fixed value of b_1 we split the sum over ν into two parts $Q(n)$ and $R(n)$, depending on B and therefore b_1, such that in $Q(n)$, $(2\nu - B/6p) < 0$ and in $R(n)$, $(2\nu - B/6p) \geqq 0$. Using Rademacher's argument (§7 in [8]) and utilizing Theorem 2 we find that

$$R(n) = O(e^{2\pi n N^{-2}} n^{1/3} N^{-1/3+\epsilon}). \tag{8.3}$$

Now $Q(n)$ contains all ν such that $\nu < B/12p$, so that if $B \leqq 0$ the sum over ν is vacuous and $Q(n) = 0$. It is of interest to investigate under what conditions $B \leqq 0$. By (3.14) $B = rp^2 - 6p\sum_{i=1}^r b_i + 6\sum_{i=1}^r b_i^2$. Treating B as a function of r continuous variables we see that $\partial B/\partial b_i = 12b_i - 6p < 0$, since $b_i \leqq (p-1)/2$. Therefore, B assumes its maximal value if the elements of b are $1, 2, \cdots, r$. We conclude that

$$B \leqq rp^2 - 6p\sum_{i=1}^r i + 6\sum_{i=1}^r i^2$$
$$= 2r^3 - 3(p-1)r^2 + (p^2 - 3p + 1)r.$$

This implies that if $2r^2 - 3(p-1)r + (p^2 - 3p + 1) \leqq 0$ then $B \leqq 0$ for all b_1. Applying the quadratic formula we see that this condition is equivalent to

$$\frac{3(p-1) - (p^2 + 6p + 1)^{1/2}}{4} \leqq r \leqq \frac{3(p-1) + (p^2 + 6p + 1)^{1/2}}{4}.$$

Since the right-hand inequality is always satisfied and since

$$\frac{3(p-1) - (p^2 + 6p + 1)^{1/2}}{4} < \frac{p-1}{2}$$

we see that $B \leqq 0$ for all b_1 if $r = (p-1)/2$.

Let us now assume that $r < (p-1)/2$ and that b consists of the r consecutive integers $x, x+1, \cdots, x+r-1$. Then

$$B = (6x^2 - 6(p - r + 1)x + (p^2 - 3pr + 3p) + (2r^2 - 3r + 1))r.$$

Making use of the quadratic formula we deduce that $B \leqq 0$ if

$$\frac{p - r + 1}{2} - \frac{(p^2 - r^2 + 1)^{1/2}}{(12)^{1/2}} \leqq x \leqq \frac{p - r + 1}{2} + \frac{(p^2 - r^2 + 1)^{1/2}}{(12)^{1/2}}.$$

Since

$$\frac{p-r+1}{2}+\frac{(p^2-r^2+1)^{1/2}}{(12)^{1/2}} \geqq \frac{p}{2}$$

we see that the right-hand inequality always holds. Since B is a decreasing function of each b_i we conclude that

$$\text{if } \min b_i \geqq \frac{p-r+1}{2}-\frac{(p^2-r^2+1)^{1/2}}{2(3)^{1/2}} \quad \text{then} \quad B \leqq 0.$$

Conversely,

$$\text{if } \max b_i < \frac{p+r-1}{2}-\frac{(p^2-r^2+1)^{1/2}}{2(3)^{1/2}} \quad \text{then} \quad B > 0.$$

If $r=1$ the last inequality reduces to Livingood's result that $Q(n)$ is different from 0 only for $b=b_1$ such that $1 \leqq b < p(3-3^{1/2})/6$.

In evaluating $Q(n)$ we again utilize Rademacher's technique as found in [8] and obtain

$$(8.4) \qquad Q(n) = 2\pi \sum_{k=1;k\equiv 0(p)}^{N} \sum_{\nu<B/12p} p_b(\nu)A_{k,b}(n,\nu)L_{k,b}(n,\nu) + O(e^{2\pi n N^{-2}} n^{1/3} N^{-1/3+\epsilon}).$$

Here

$$(8.5) \qquad A_{k,b}(n,\nu) = \sum_{h \bmod k}{}' \omega_a(h,k) \exp\{-2\pi i(nh-\nu h')/k\}$$

where $h \equiv \pm b_1\beta \pmod p$, and

$$L_{k,b}(n,\nu) = \frac{1}{2\pi i}\int_R \sum_{\mu=0}^{\infty} \frac{((B-12\nu p)\pi/6pk^2w)^{\mu}}{\mu!} \sum_{\lambda=0}^{\infty} \frac{((12np-A)\pi w/6p)^{\lambda}}{\lambda!}\, dw$$

where R is a rectangle containing the origin.

By Cauchy's Residue Theorem we see that

$$(8.6) \quad L_{k,b}(n,\nu) = \begin{cases} \dfrac{(B-12\nu p)^{1/2}}{k(12np-A)^{1/2}} I_1\{\pi(12np-A)^{1/2}(B-12\nu p)^{1/2}/3pk\} & \text{if } n > \dfrac{A}{12p}, \\[2ex] \dfrac{(B-12\nu p)\pi}{6pk^2} & \text{if } n = \dfrac{A}{12p}, \end{cases}$$

where $I_1(z)$ is the Bessel function of the first order with purely imaginary argument.

The possibility that $A/12p$ be an integer is one which does not occur in the investigations of Lehner and Livingood. As an example to illustrate the possibility of this occurrence in the general case, if we take $p=1327$, $r=12$, $a=\{5, 6, 7, 8, 10, 12, 14, 16, 18, 20, 22, 24\}$, we find that $A/12p=n=1248$.

The determination of $L_{k,b}(n, \nu)$ in case $n<A/12p$ is possible but of no significance here since, as we shall see, the final formula for $p_a(n)$ will require a knowledge of $p_a(\nu)$ for $\nu<A/12p$ if $A>0$.

We now take up the study of $p_a^{(2)}(n)$. From (8.1), (8.2), and (4.19) of Theorem 1

$$p_a^{(2)}(n) = \frac{1}{2^r}\sum_{h,k}{}' \left(\prod_{i=1}^{r} \csc \pi\alpha_i/p\right)\chi_a(h, k) \exp\{-2\pi inh/k\} \cdot \int_{-\theta'}^{\theta''} \sum_{\nu=0}^{\infty} c_a^{(k)}(\nu) \exp\{2\pi iH'\nu/k\} \cdot\exp\{-(\pi/Kkw)(2\pi - r/6) + \pi w(2n - A/6p)\}d\phi$$

where $0\leqq h<k\leqq N$, $(p, k)=1$.

We split the sum over ν into two parts $Q^*(n)$ and $R^*(n)$ such that in $Q^*(n)$, $(2\nu-r/6)<0$ and in $R^*(n)$, $(2\nu-r/6)\geqq 0$. Since $(\prod_{i=1}^{r} \csc \pi\alpha_i/p) \leqq \csc^r \pi/p$ we see by exactly the same argument as in the case of $R(n)$, recalling the remark following (4.13), and employing Theorem 3, that

$$R^*(n) = O(e^{2\pi nN^{-2}} n^{1/3}N^{-1/3+\epsilon}). \tag{8.7}$$

$Q^*(n)$ contains all ν such that $\nu<r/12$. Using the same procedure as in the case of $Q(n)$ we find that

$$Q^*(n) = \frac{2\pi}{2^r}\sum_{k=1;\,(k,p)=1}^{N} \left(\prod_{i=1}^{r} \csc \pi\alpha_i/p\right) \sum_{\nu<r/12} c_a^{(k)}(\nu)B_k(n, \nu)L_k^*(n, \nu) + O(e^{2\pi nN^{-2}} n^{1/3}N^{-1/3+\epsilon}), \tag{8.8}$$

where

$$B_k(n, \nu) = \sum_{h \bmod k}{}' \chi_a(h, k) \exp\{-2\pi i(nh - \nu H')/k\} \tag{8.9}$$

and

$$L_k^*(n,\nu) = \begin{cases} \dfrac{(r - 12\nu)^{1/2}}{k(12np - A)^{1/2}} I_1\{\pi(12np - A)^{1/2}(r - 12\nu)^{1/2}/3pk\} & \text{if } n > \dfrac{A}{12p}, \\[2ex] \dfrac{(r - 12\nu)\pi}{6pk^2} & \text{if } n = \dfrac{A}{12p}. \end{cases} \tag{8.10}$$

Since $p_a^{(1)}(n)=\sum_{b_1}(Q(n)+R(n))$ and $p_a^{(2)}(n)=Q^*(n)+R^*(n)$ we have, letting $N\to\infty$ in (8.3), (8.4), (8.7), and (8.8)

THEOREM 4. *The number, $p_a(n)$, of partitions of a positive integer n, $n\geqq A/12p$, into positive summands of the form $pm\pm a_i$, $a_i\in a$, is given by the convergent series*

$$
\begin{aligned}
p_a(n) = 2\pi &\sum_{k=1;\,k\equiv 0(p)}^{\infty}\ \sum_{b_1=1}^{(p-1)/2}\ \sum_{\nu<B/12p} p_b(\nu)A_{k,b}(n,\nu)L_{k,b}(n,\nu) \\
&+\frac{\pi}{2^{r-1}}\sum_{k=1;\,(p,k)=1}^{\infty}\left(\prod_{i=1}^{r}\csc \pi\alpha_i/p\right)\sum_{\nu<r/12} c_a^{(k)}(\nu)B_k(n,\nu)L_k^*(n,\nu)
\end{aligned}
\tag{8.11}
$$

where $A_{k,b}(n,\nu)$, $L_{k,b}(n,\nu)$, $B_k(n,\nu)$, $L_k^*(n,\nu)$ are given by (8.5), (8.6), (8.9), *and* (8.10) *respectively.*

We remark that this result, except for the case $n=A/12p$, is contained in a more general theorem due to Petersson (Satz 13 in [5]). In form, however, Petersson's statement is quite different from (8.11).

V. AN ADDITION TO THE SET a

9. Our next objective(4) is to modify Theorem 4 so that in the partitions of n, summands congruent to 0 modulo p are admissible. We first study the function

$$F_p(x) = \prod_{m=1}^{\infty}(1-x^{pm})^{-1} = F(x^p) = 1+\sum_{n=1}^{\infty}p_p(n)x^n$$

where $F(x)$ is the generating function of the Euler partition function. $p_p(n)$ represents the number of partitions of n such that every summand is congruent to 0 (mod p). The transformation equation of $F_p(x)$ is easily obtained from that of $F(x)$ ((2.4) in [7]).

LEMMA 1. *$F_p(x)$ satisfies the transformation equation*

$$
\begin{aligned}
F_p(\exp\{2\pi ih/k - 2\pi z/k\}) &= z^{1/2}\omega_p(h,k)\exp\left\{\frac{\pi}{6pk}(B_p/z - A_pz)\right\} \\
&\qquad\cdot F_p(\exp\{2\pi ih'/k - 2\pi/kz\})
\end{aligned}
$$

if $p\mid k$, and the equation

$$
\begin{aligned}
F_p(\exp\{2\pi ih/k - 2\pi z/k\}) &= z^{1/2}p^{1/2}\chi_p(h,k) \\
&\qquad\cdot\exp\left\{\frac{\pi}{6pk}(1/2z - A_pz)\right\}F(\exp\{2\pi iH'/k - 2\pi/Kz\})
\end{aligned}
$$

if $p\nmid k$. Here

---

(4) The proof of Lemma 1 in this section was suggested by the referee.

$$A_p = B_p = p^2/2,$$

$$\omega_p(h, k) = \exp\left\{\pi i \sum_\mu ((\mu/k))((h\mu/k)), \quad where \quad \mu = p, 2p, \cdots, k,\right.$$

$$\chi_p(h, k) = \exp\left\{\pi i \sum_\mu ((\mu/K))((h\mu/k)), \quad where \quad \mu = p, 2p, \cdots, K.\right.$$

10. Defining the set $\bar{a} = \{a_1, a_2, \cdots, a_r, p\}$ we now wish to determine a convergent series for $p_{\bar{a}}(n)$, the number of partitions of n into summands congruent to elements of $\bar{a}$ or their negatives modulo p. The general procedure parallels that used in our investigation of $p_a(n)$. Theorem 1 and Lemma 1 yield the required transformation equations, and the Farey dissection method is again utilized for the integration. The main differences arise from the fact that the presence of $z^{1/2}$ necessitates the introduction of a loop integral and that the trivial estimate $O(k)$ replaces those of Theorems 2 and 3. Our method is similar to that employed by Rademacher and Zuckerman [9].

Consider the generating function

$$F_{\bar{a}}(x) = F_p(x)F_a(x) = 1 + \sum_{n=1}^{\infty} p_{\bar{a}}(n)x^n, \tag{10.1}$$

which is convergent in the interior of the unit circle. By Theorem 1, Lemma 1, and multiplication we have

THEOREM 5. *$F_{\bar{a}}(x)$ satisfies the transformation equation*

$$\begin{aligned} F_{\bar{a}}(\exp\{2\pi ih/k - 2\pi z/k\}) &= z^{1/2}\omega_{\bar{a}}(h, k) \exp\left\{\frac{\pi}{6pk}(\bar{B}/z - \bar{A}z)\right\} \\ &\quad \cdot F_{\bar{b}}(\exp\{2\pi ih'/k - 2\pi/kz\}) \end{aligned} \tag{10.2}$$

if $p \mid k$, and the equation

$$\begin{aligned} F_{\bar{a}}(\exp\{2\pi ih/k - 2\pi z/k\}) &= \frac{z^{1/2}p^{1/2}}{2^r}\chi_{\bar{a}}(h, k) \prod_{i=1}^{r} \csc \pi\alpha_i/p \\ &\quad \cdot \exp\left\{\frac{\pi}{6pk}(\bar{r}/z - \bar{A}z)\right\} H_{\bar{a}}(\exp\{2\pi iH'/k - 2\pi/Kz\}) \end{aligned} \tag{10.3}$$

if $p \nmid k$. Here

$$\omega_{\bar{a}}(h, k) = \omega_a(h, k)\omega_p(h, k), \tag{10.4}$$

$$\chi_{\bar{a}}(h, k) = \chi_a(h, k)\chi_p(h, k), \tag{10.5}$$

$$\bar{A} = A + A_p, \tag{10.6}$$

$$\bar{B} = B + B_p, \tag{10.7}$$

$$\bar{r} = r + 1/2, \tag{10.8}$$

$$\bar{b} = \{b_1, b_2, \cdots, b_r, p\}, \tag{10.9}$$

(10.10) $$H_{\bar{a}}(x) = H_a(x)F(x) = 1 + \sum_{\nu=1}^{\infty} \overset{(k)}{c_{\bar{a}}}(\nu)x^{\nu}.$$

By Cauchy's integral formula and (10.1)

$$p_{\bar{a}}(n) = \frac{1}{2\pi i}\int_C \frac{F_{\bar{a}}(x)}{x^{n+1}}\,dx \quad \sum_{h,k;\,0\le h<k\le N}{}' \frac{1}{2\pi i}\int_{\xi_{h,k}} \frac{F_{\bar{a}}(x)}{x^{n+1}}\,dx.$$

Defining ϕ, w, z as before we have

(10.11) $$p_{\bar{a}}(n) = \sum_{h,k;\,0\le h<k\le N}{}' \exp\{-2\pi inh/k\}\int_{-\theta'}^{\theta''} F_{\bar{a}} \exp\left\{\frac{2\pi ih}{k} - \frac{2\pi z}{k}\right\} \cdot \exp\{2\pi nw\}\,d\phi.$$

(10.12) $$p_{\bar{a}}(n) = \overset{(1)}{p_{\bar{a}}}(n) + \overset{(2)}{p_{\bar{a}}}(n)$$

where $p_{\bar{a}}^{(1)}(n)$ is the sum of the terms for which $p\,|\,k$ and $p_{\bar{a}}^{(2)}(n)$ is the sum of the terms for which $p\nmid k$.

Selecting β so that $a_1\beta\equiv 1 \pmod p$ we have by (10.2)

$$\begin{aligned}\overset{(1)}{p_{\bar{a}}}(n) = \sum_{b_1=1}^{(p-1)/2} \exp\{2\pi nN^{-2}\} \sum_{h,k}{}' \omega_{\bar{a}}(h,k)\exp\{-2\pi inh/k\}\\ \cdot\int_{-\theta'}^{\theta''}\exp\{-2\pi in\phi\}z^{1/2}\exp\left\{\frac{\pi}{6pk}(\overline{B}/z - \overline{A}z)\right\}\\ \cdot\sum_{\nu=0}^{\infty} p_{\bar{b}}(\nu)\exp\{2\pi i\nu h'/k - 2\pi\nu/kz\}\,d\phi\end{aligned}$$

where $0\le h<k\le N$, $h\equiv \pm b_1\beta \pmod p$, $k\equiv 0 \pmod p$.

We now restrict our attention to values of n such that $n>\overline{A}/12p$. There is no real loss in this restriction since our final formula for $p_{\bar{a}}(n)$ will require prior knowledge of the values of $p_{\bar{a}}(\nu)$ for $\nu<\overline{A}/12p$. $n=\overline{A}/12p$ is impossible since by (10.6) $\overline{A}$ is not an integer.

For a fixed value of b_1 we split the sum over ν into two parts $Q(n)$ and $R(n)$ such that in $Q(n)$, $(2\nu-\overline{B}/6p)<0$ and in $R(n)$, $(2\nu-\overline{B}/6p)>0$. Following the method detailed in [9] we find that

(10.13) $$R(n) = O(e^{2\pi nN^{-2}}N^{-1/2})$$

and

(10.14) $$Q(n) = 2\pi\sum_{k=1;\,k\equiv 0\,(p)}^{N}\ \sum_{\nu<\overline{B}/12p} p_{\bar{b}}(\nu)\overline{A}_{k,b}(n,\nu)\overline{L}_{k,b}(n,\nu) + O(e^{2\pi nN^{-2}}N^{-1/2}).$$

Here

$$\overline{A}_{k,b}(n,\nu) = \sum_{h \bmod k}{}' \omega_{\bar{a}}(h,k) \exp\{-2\pi i(nh - \nu h')/k\} \tag{10.15}$$

where $h \equiv \pm b_1\beta \pmod p$, and

$$\overline{L}_{k,b}(n,\nu) = \frac{(\overline{B} - 12\nu p)^{3/4}}{k(12np - \overline{A})^{3/4}} I_{(3/2)}\{\pi(12np - \overline{A})^{1/2}(\overline{B} - 12\nu p)^{1/2}/3pk\}. \tag{10.16}$$

$I_{(3/2)}(z)$ is the Bessel function.

The treatment of $p_{\bar{a}}^{(2)}(n)$ is similar. (10.3), (10.11) yield

$$p_{\bar{a}}^{(2)}(n) = \frac{p^{1/2}}{2^r} \exp\{2\pi n N^{-2}\} \sum_{h,k}{}' \chi_{\bar{a}}(h,k) \exp\{-2\pi i nh/k\}$$
$$\cdot \prod_{i=1}^{r} \csc \pi\alpha_i/p \int_{-\theta'}^{\theta''} \exp\{-2\pi n i\phi\} z^{1/2} \exp\left\{\frac{\pi}{6pk}(\bar{r}/z - \overline{A}z)\right\}$$
$$\cdot \sum_{\nu=0}^{\infty} c_{\bar{a}}^{(k)}(\nu) \exp\{2\pi i\nu H'/k - 2\pi\nu/Kz\} d\phi$$

where $0 \leqq h < k \leqq N$, $(p, k) = 1$.

The sum over ν is split into parts $Q^*(n)$ and $R^*(n)$ such that in $Q^*(n)$, $(2\nu - \bar{r}/6) < 0$ and in $R^*(n)$, $(2\nu - \bar{r}/6) > 0$. Since, by the remarks following (4.13), the series for $H_{\bar{a}}(x)$ converges in the unit circle uniformly in k, and since $\prod_{i=1}^{r} \csc \pi\alpha_i/p \leqq \csc^r \pi/p$ we have by exactly the same procedure as in the case $p \mid k$

$$R^*(n) = O(e^{2\pi n N^{-2}} N^{-1/2}) \tag{10.17}$$

and

$$Q^*(n) = \frac{\pi p^{1/2}}{2^{r-1}} \sum_{k=1;\,(p,k)=1}^{N} \left(\prod_{i=1}^{r} \csc \pi\alpha_i/p\right) \sum_{\nu<\bar{r}/12} c_{\bar{a}}^{(k)}(\nu) \overline{B}_k(n,\nu) \overline{L}_k^*(n,\nu) + O(e^{2\pi n N^{-2}} N^{-1/2}) \tag{10.18}$$

where

$$\overline{B}_k(n,\nu) = \sum_{h \bmod k}{}' \chi_{\bar{a}}(h,k) \exp\{-2\pi i(nh - \nu H')/k\} \tag{10.19}$$

and

$$\overline{L}_k^*(n,\nu) = \frac{(\bar{r} - 12\nu)^{3/4}}{k(12np - \overline{A})^{3/4}} I_{(3/2)}\{\pi(12np - \overline{A})^{1/2}(\bar{r} - 12\nu)^{1/2}/3pk\}. \tag{10.20}$$

Summing $R(n)$ and $Q(n)$ over b_1, and letting $N \to \infty$ we have by (10.13), (10.14), (10.17), (10.18)

THEOREM 6. *The number, $p_{\bar{a}}(n)$, of partitions of a positive integer n, $n>\bar{A}/12p$, into positive summands congruent modulo p to elements of $\bar{a}$ or their negatives is given by the convergent series*

$$(10.21)\quad \begin{aligned} p_{\bar{a}}(n) &= 2\pi \sum_{k=1;k\equiv 0\,(p)}^{\infty} \sum_{b_1=1}^{(p-1)/2} \sum_{\nu<\bar{B}/12p} p_{\bar{b}}(\nu)\bar{A}_{k,b}(n,\nu)\bar{L}_{k,b}(n,\nu) \\ &\quad + \frac{\pi p^{1/2}}{2^{r-1}} \sum_{k=1;\,(p,k)=1}^{\infty} \left(\prod_{i=1}^{r} \csc \pi\alpha_i/p\right) \sum_{\nu<\bar{r}/12} c_{\bar{a}}^{(k)}(\nu)\bar{B}_k(n,\nu)\bar{L}_k^*(n,\nu) \end{aligned}$$

where $\bar{A}_{k,b}(n, \nu)$, $\bar{L}_{k,b}(n, \nu)$, $\bar{B}_k(n, \nu)$, $\bar{L}_k^*(n, \nu)$ *are given by* (10.15), (10.16), (10.19), (10.20) *respectively.*

VI. ASYMPTOTIC FORMULAS

11. As the series for $p_a(n)$ and $p_{\bar{a}}(n)$, as developed in Theorems 4 and 6, are extremely complicated it is worthwhile to investigate the possibility of obtaining asymptotic formulas of a simpler form in order to approximate them for large n. We first consider $p_a(n)$ where $n>A/12p$. As we shall see, $k=1$ yields the dominant term in the expansion. When $k=1$ it follows from (4.8) that $\alpha_i=a_i$ for $i=1, 2, \cdots, r$. We also note that $B_1(n, \nu)=1$. Writing

$$(11.1)\quad G(\nu) = (r - 12\nu)^{1/2},$$

$$(11.2)\quad T = \frac{\pi(12np - A)^{1/2}}{3p},$$

$$(11.3)\quad W = \sum_{\nu<r/12} c_a^{(1)}(\nu)G(\nu)I_1\{TG(\nu)\},$$

we have by (8.11)

$$(11.4)\quad p_a(n) = \frac{\pi}{2^{r-1}}\left(\prod_{i=1}^{r} \csc \pi a_i/p\right)\frac{W}{(12np - A)^{1/2}}\{1 + S_1 + S_2\}$$

where

$$(11.5)\quad \begin{aligned} S_1 &= 2^r\left(\prod_{i=1}^{r} \sin \pi a_i/p\right) \sum_{k=p;\,k\equiv 0(p)}^{\infty} \sum_{b_1} \sum_{\nu<B/12p} p_b(\nu)A_{k,b}(n,\nu) \\ &\quad \cdot \frac{(B - 12\nu p)^{1/2}}{k} \frac{I_1\{T(B - 12\nu p)^{1/2}/k\}}{W} \end{aligned}$$

and

$$S_2 = \left(\prod_{i=1}^{r} \sin \pi a_i/p\right) \sum_{k=2;\,(k,p)=1}^{\infty} \left(\prod_{i=1}^{r} \csc \pi a_i/p\right) \sum_{\nu<r/12} c_a^{(k)}(\nu) B_k(n,\nu) \frac{G(\nu)}{k} \cdot \frac{I_1\{TG(\nu)/k\}}{W}. \tag{11.6}$$

We first investigate the magnitude of S_1. If $B \leqq 0$ for all b_1 then $S_1 = 0$. Otherwise, as we have shown in §8,

$$B \leqq rp^2 - 6p \sum_{i=1}^{r} i + 6 \sum_{i=1}^{r} i^2 = M \quad \text{where} \quad M > 0. \tag{11.7}$$

Obviously, there is a constant J such that for all b_1 and $\nu < M/12p$

$$0 \leqq p_b(\nu) \leqq J. \tag{11.8}$$

From (8.5) and Theorem 2

$$A_{k,b}(n,\nu) = O(n^{1/3}k^{2/3+\epsilon}) \tag{11.9}$$

uniformly in ν and b.

Obviously

$$0 < \prod_{i=1}^{r} \sin \pi a_i/p < 1. \tag{11.10}$$

From the theory of Bessel functions

$$I_1(z) = \frac{e^z}{(2\pi z)^{1/2}}(1 + O(z^{-1})) \text{ as } z \to \infty, \tag{11.11}$$

and

$$I_1(z) = O(z) \qquad \text{if } |z| < 1. \tag{11.12}$$

Finally

$$W \sim r^{1/2} I_1\{Tr^{1/2}\}. \tag{11.13}$$

Proof. If $r \leqq 12$ there is nothing to prove. We assume, therefore, that $r > 12$. From (11.3)

$$W = r^{1/2} I_1\{Tr^{1/2}\} \cdot \left(1 + \sum_{0<\nu<r/12} c_a^{(1)}(\nu) \frac{G(\nu)}{r^{1/2}} \frac{I_1\{TG(\nu)\}}{I_1\{Tr^{1/2}\}}\right). \tag{11.14}$$

For large n it follows from (11.11) and (11.2) that

$$\frac{I_1\{TG(\nu)\}}{I_1\{Tr^{1/2}\}} = O(\exp\{T(G(1) - r^{1/2})\})$$

$$(11.15) \qquad = O\left(\exp\left\{\frac{-2\pi(nr)^{1/2}}{(3p)^{1/2}}(1 - A/12np)^{1/2}(1 - (1 - 12/r)^{1/2})\right\}\right).$$

Since $|c_a^{(1)}(\nu)|$ is bounded for $0<\nu<r/12$ the desired result follows from (11.14) and (11.15).

From (11.5), (11.7), (11.8), (11.9), (11.10), (11.13) we have

$$(11.16) \qquad S_1 = O\left(\sum_{k=p}^{\infty} n^{1/3}k^{-1/3+\epsilon}\,\frac{I_1\{TM^{1/2}/k\}}{I_1\{Tr^{1/2}\}}\right).$$

We now split the sum in (11.16) into two parts according to whether $k<TM^{1/2}=D$ or $k>D$. Utilizing (11.11) in the first case and (11.12) in the second we have

$$(11.17) \qquad \begin{aligned} S_1 = {} & O\left(\sum_{k=p}^{D} n^{1/3}k^{-1/3+\epsilon}k^{1/2}\exp\{T(M^{1/2}/k - r^{1/2})\}\right) \\ & + O\left(\sum_{k>D} n^{1/3}k^{-1/3+\epsilon}k^{-1}T^{3/2}\exp\{-Tr^{1/2}\}\right). \end{aligned}$$

We note that

$$(11.18) \qquad \frac{M^{1/2}}{k} - r^{1/2} < 0 \qquad \text{if } k \geqq p.$$

For

$$\frac{M}{k^2} \leqq \frac{M}{p^2} = \frac{1}{p^2}\left\{rp^2 - 6p\sum_{i=1}^{r} i + 6\sum_{i=1}^{r} i^2\right\} = r - 6\sum_{i=1}^{r}\frac{i}{p}(1 - i/p) < r.$$

Also, from (11.2)

$$(11.19) \qquad T \leqq \frac{\pi(12p + |A|)^{1/2}}{3p}\,n^{1/2} = \delta n^{1/2},$$

$$(11.20) \qquad T = \frac{2\pi n^{1/2}}{(3p)^{1/2}}(1 - A/12np)^{1/2} \geqq \frac{2\pi}{(3p)^{1/2}}(1 - \epsilon)n^{1/2} = \delta^* n^{1/2}$$

for $\epsilon>0$ and large n.

From (11.17)–(11.20) we conclude that

$$\begin{aligned} S_1 = {} & O(n^{1/3}n^{7/12+\epsilon}\exp\{-d\delta^* n^{1/2}\}) \\ & + O(n^{1/3}n^{3/4}n^{-1/6+\epsilon}\exp\{-r^{1/2}\delta^* n^{1/2}\}) \end{aligned}$$

where

(11.21) $$d = r^{1/2} - M^{1/2}/p.$$

Therefore, using (11.20) we have

$$S_1 = O(n \cdot \exp\{-d\delta^* n^{1/2}\})$$

(11.22) $$= O\left(\exp\left\{-\left(\frac{2\pi d}{(3p)^{1/2}} - \epsilon\right) n^{1/2}\right\}\right).$$

The study of S_2 is quite similar. We remark first that $|c_a^{(k)}(\nu)|$ is bounded uniformly in k and ν for $\nu < r/12$. For $c_a^{(k)}(\nu)$ has exactly the same form for every value of k. For example,

$$c_a^{(k)}(1) = \sum_{i=1}^{r} (\rho_i + \bar{\rho}_i).$$

Furthermore, for each k, $|\rho_i| = 1$ for $i = 1, 2, \cdots, r$.

From (8.9) and Theorem 3 $B_k(n, \nu) = O(n^{1/3} k^{2/3+\epsilon})$ uniformly in ν.

Making use of these remarks and proceeding as before we find that

$$S_2 = O(n \cdot \exp\{-r^{1/2}\delta^* n^{1/2}/2\})$$

(11.23) $$= O\left(\exp\left\{-\left(\frac{\pi r^{1/2}}{(3p)^{1/2}} - \epsilon\right) n^{1/2}\right\}\right).$$

From (11.4), (11.22), (11.23) we obtain

THEOREM 7. *As* $n \to \infty$

(11.24) $$p_a(n) = \frac{\pi}{2^{r-1}}\left(\prod_{i=1}^{r} \csc \pi a_i/p\right) \frac{W}{(12np - A)^{1/2}} \cdot \left(1 + O\left(\exp\left\{-\left(\frac{2\pi c r^{1/2}}{(3p)^{1/2}} - \epsilon\right) n^{1/2}\right\}\right)\right)$$

where

(11.25) $$c = \min\{1/2, 1 - M^{1/2}/pr^{1/2}\}.$$

By (11.14) and (11.15) we have also:

COROLLARY 7.1. *As* $n \to \infty$

(11.26) $$p_a(n) = \frac{\pi r^{1/2}}{2^{r-1}}\left(\prod_{i=1}^{r} \csc \pi a_i/p\right) \frac{I_1\{Tr^{1/2}\}}{(12np - A)^{1/2}} \cdot \left(1 + O\left(\exp\left\{-\left(\frac{2\pi c r^{1/2}}{(3p)^{1/2}} - \epsilon\right) n^{1/2}\right\}\right)\right)$$

where c is given by (11.25) *if* $r \leqq 12$ *and*

$$c = \min\{1/2,\ 1 - M^{1/2}/pr^{1/2},\ 1 - (1 - 12/r)^{1/2}\}$$

otherwise.

This result is easily identified with those of Petersson ((10.8) in [5]) and Grosswald (17‴ in [1]).

If we use (11.11) we have from (11.2) and (11.26)

COROLLARY 7.2. *As* $n \to \infty$

$$(11.27)\quad p_a(n) = \frac{(6p)^{1/2} r^{1/4}}{2^r(12np - A)^{3/4}} \left(\prod_{i=1}^{r} \csc \pi a_i/p \right) \exp\{Tr^{1/2}\} \cdot \{1 + O(n^{-1/2})\}.$$

This is essentially Petersson's result ((10.9) in [5]).

12. Turning now to $p_{\bar a}(n)$ we find that as in the case of $p_a(n)$ the dominant term in the expansion is given by $k=1$. Defining

$$(12.1)\qquad \overline{G}(\nu) = (\bar r - 12\nu)^{1/2},$$

$$(12.2)\qquad \overline{T} = \frac{\pi(12np - \overline{A})^{1/2}}{3p},$$

$$(12.3)\qquad \overline{W} = \sum_{\nu < \bar r/12} c_{\bar a}^{(1)}(\nu)(\bar r - 12\nu)^{3/4} I_{3/2}\{\overline{T}\overline{G}(\nu)\},$$

$$(12.4)\qquad \overline{M} = M + p^2/2,$$

we have

THEOREM 8. *As* $n \to \infty$

$$(12.5)\qquad \begin{aligned} p_{\bar a}(n) = {} & \frac{\pi p^{1/2}}{2^{r-1}} \left(\prod_{i=1}^{r} \csc \pi a_i/p \right) \frac{\overline{W}}{(12np - \overline{A})^{3/4}} \\ & \cdot \left(1 + O\left(\exp\left\{-\left(\frac{2\pi c \bar r^{1/2}}{(3p)^{1/2}} - \epsilon\right) n^{1/2}\right\}\right)\right) \end{aligned}$$

where

$$(12.6)\qquad c = \min\{1/2,\ 1 - \overline{M}^{1/2}/\bar r^{1/2} p\}.$$

COROLLARY 8.1. *As* $n \to \infty$

$$(12.7)\qquad \begin{aligned} p_{\bar a}(n) = {} & \frac{\pi p^{1/2} \bar r^{3/4}}{2^{r-1}} \left(\prod_{i=1}^{r} \csc \pi a_i/p \right) \frac{I_{3/2}\{\overline{T}\bar r^{1/2}\}}{(12np - \overline{A})^{3/4}} \\ & \cdot \left(1 + O\left(\exp\left\{-\left(\frac{2\pi c \bar r^{1/2}}{(3p)^{1/2}} - \epsilon\right) n^{1/2}\right\}\right)\right) \end{aligned}$$

where c is given by (12.6) *if* $\bar r < 12$ *and*

$$c = \min\{1/2,\ 1 - \overline{M}^{1/2}/p\bar{r}^{1/2},\ 1 - (1 - 12/\bar{r})^{1/2}\}\ \textit{otherwise.}$$

Since $I_{3/2}(z) = e^z/(2\pi z)^{1/2}(1+O(z^{-1}))$ as $z \to \infty$ we have from (12.7) and (12.2)

COROLLARY 8.2. *As* $n \to \infty$

$$(12.8)\qquad p_{\bar{a}}(n) = \frac{p(6\bar{r})^{1/2}}{2^r(12np - \overline{A})}\left(\prod_{i=1}^{r} \csc \pi a_i/p\right)\exp\{\overline{T}\bar{r}^{1/2}\}(1 + O(n^{-1/2})).$$

Since the proofs of these results parallel those for $p_a(n)$ they are omitted.

VII. SOME SPECIAL CASES

13. It is of interest to apply the results obtained above to some particular sets a and to a restricted range of values for r. If we take

$$a = \{1, 2, \cdots, (p-1)/2\} \quad \text{then} \quad r = (p-1)/2$$

and we find that

$$(13.1)\qquad B = A = \frac{p - p^2}{2} < 0 \qquad \text{for all } b_1.$$

Also,

$$(13.2)\qquad 2^{-r}\prod_{i=1}^{r} \csc \pi a_i/p = \left(\prod_{i=1}^{(p-1)/2} 2 \sin \pi a_i/p\right)^{-1} = p^{-1/2}.$$

By (13.1), (13.2), (8.11) we have

THEOREM 9. *If $p(n; \bar{p})$ is the number of partitions of n such that no summand is divisible by p then*

$$(13.3)\qquad p(n; \bar{p}) = \frac{2\pi}{(p)^{1/2}} \sum_{k=1;\,(p,k)=1}^{\infty} \sum_{\nu<(p-1)/24} c_a^{(k)}(\nu) B_k(n, \nu) L_k^*(n, \nu)$$

where $B_k(n, \nu)$ and $L_k^*(n, \nu)$ are given by (8.9) *and* (8.10) *respectively.*

This result can be given a different interpretation. By a theorem of Tietze [13], if k is a positive integer then the number of partitions of n in which no summand is divisible by k is equal to the number of partitions of n in which no summand is repeated k or more times. Thus, (13.3) furnishes a convergent series for the number of partitions of n in which no summand appears more than $(p-1)$ times. We also note that (13.3) agrees with a result due to Petersson (equations (2.7)–(2.11) in [6]) if k_0 and k_1 are set equal to 1 and 0 respectively in the latter.

14. When $r \leqq 12$ (which is always true if $p \leqq 23$) only $c_a^{(k)}(0) = 1$ appears in

(8.11). Then

$$p_a(n) = 2\pi \sum_{k=1;k\equiv 0(p)}^{\infty} \sum_{b_1=1}^{(p-1)/2} \sum_{\nu<B/12p} p_b(\nu)A_{k,b}(n,\nu)L_{k,b}(n,\nu) \tag{14.1}$$
$$+ \frac{\pi}{2^{r-1}} \sum_{k=1;\,(p,k)=1}^{\infty} \left(\prod_{i=1}^{r} \csc \pi\alpha_i/p\right) B_k(n,0)L_k^*(n,0).$$

For $r=1$ this formula reduces to that obtained by Livingood ((7.9) in [4]).

Now, the maximum value of B is obtained when $b=\{1, 2, \cdots, r\}$. In this case

$$B = rp^2 - 3(r^2+r)p + 2r^3 + 3r^2 + r.$$

Therefore, for all b_1 such that $B>0$, if

$$\frac{rp^2 - 3(r^2+r)p + 2r^3 + 3r^2 + r}{12p} < 1$$

then only $p_b(0)=1$ appears in (14.1). Using the quadratic formula we find that the last inequality is equivalent to

$$p < \frac{3(r^2+r+4) + (r^4+6r^3+77r^2+72r+144)^{1/2}}{2r}. \tag{14.2}$$

We conclude that if $r \leqq 12$ and p satisfies (14.2) then

$$p_a(n) = 2\pi \sum_{k=1;k\equiv 0(p)}^{\infty} \sum_{b_1\ni B>0} A_{k,b}(n,0)L_{k,b}(n,0) \tag{14.3}$$
$$+ \frac{\pi}{2^{r-1}} \sum_{k=1;\,(p,k)=1}^{\infty} \left(\prod_{i=1}^{r} \csc \pi\alpha_i/p\right) B_k(n,0)L_k^*(n,0).$$

We give the maximum value of p for which (14.3) holds for some special values of r. For $r=1, 2, 3, 6, 9, 12$, $p=17, 13, 13, 17, 23, 23$ respectively.

15. If $p\equiv 1 \pmod 4$ then the set $a=\{a_1, a_2, \cdots, a_r\}$ can be taken so that in conjunction with $\{p-a_1, p-a_2, \cdots, p-a_r\}$ we have either the set of quadratic residues or the set of quadratic nonresidues modulo p. In this case $r=(p-1)/4$. If we consider first the case of the quadratic residues we obtain

$$A = A^+ = \sum_{i=1}^{r} (p^2 - 6pa_i + 6a_i^2) = \frac{1}{2} \sum_{(j/p)=+1} (p^2 - 6pj + 6j^2) \tag{15.1}$$

since $p^2-6pj+j^2=p^2-6p(p-j)+6(p-j)^2$. Here (j/p) stands for the Legendre symbol. Noting that in the last sum of (15.1) μ appears if, and only if, $p-\mu$ appears we have

(15.2) $$A^+ = -\frac{1}{2}p^2(p-1) + 3\sum_{(j/p)=+1} j^2.$$

Similarly, in the case of quadratic nonresidues we have

(15.3) $$A = A^- = -\frac{1}{2}p^2(p-1) + 3\sum_{(j/p)=-1} j^2.$$

Now let

(15.4) $$Q^+ = (n - A^+/12p)^{1/2}, \qquad Q^- = (n - A^-/12p)^{1/2},$$

(15.5) $$q = \pi\left(\frac{p-1}{3p}\right)^{1/2}.$$

Letting $p^+(n;p)$ and $p^-(n;p)$ denote the number of partitions of n into quadratic residues and nonresidues (mod p) respectively, we have from (11.26), (11.2) for sufficiently large n

(15.6) $$p^\pm(n;p) = \frac{q}{2^{(p+3)/4}}\left(\prod_{(j/p)=\pm 1} \sin \pi j/p\right)^{-1/2} (Q^\pm)^{-1} I_1\{qQ^\pm\} \cdot\{1 + O(e^{-\delta n^{1/2}})\}$$

where $\delta > 0$. Here either $+$ or $-$ is to be taken throughout in both members.

From the theory of Bessel functions we have

(15.7) $$I_1\{qQ^\pm\} = \frac{\exp\{qQ^\pm\}}{(2\pi qQ^\pm)^{1/2}}\{1 - 3/8qQ^\pm + O(n^{-1})\}.$$

From (15.6), (15.7), and division we have for large n

(15.8) $$\frac{p^+(n;p)}{p^-(n;p)} = \left(\prod_{j=1}^{p-1} (\sin \pi j/p)^{(j/p)}\right)^{-1/2} \cdot\left(1 - \frac{q}{2}(A^+/12p - A^-/12p)n^{-1/2} + O(n^{-1})\right).$$

By a theorem of Dirichlet

(15.9) $$\left(\prod_{j=1}^{p-1} (\sin \pi j/p)^{(j/p)}\right)^{-1/2} = \epsilon^h$$

where h is the class number of the real quadratic field $R(p^{1/2})$ and ϵ is its fundamental unit.

By (15.2) and (15.3)

(15.10) $$\frac{A^+}{12p} - \frac{A^-}{12p} = \frac{1}{4p}\sum_{j=1}^{p-1} (j/p)j^2.$$

This last sum is well known (see [12, pp. 677–678]) and is equal to $4cp\alpha_5(p)$ where $c=1/240$ if $p\equiv 1 \pmod 8$, $c=1/560$ if $p\equiv 5 \pmod 8$, and $\alpha_5(p)$ is the number of representations of p as 5 squares, representations differing by sign or order of summands being considered as distinct.

Using (15.8), (15.9), (15.10) we state

THEOREM 10. *If* $p\equiv 1 \pmod 4$ *and* $n\to\infty$ *then*

$$\frac{p^+(n;p)}{p^-(n;p)} = \epsilon^h\left(1 - \frac{\pi}{6}(3(p-1)/p)^{1/2}c\alpha_5(p)n^{-1/2} + O(n^{-1})\right). \tag{15.11}$$

This result was previously obtained by both Petersson ((5.12) in [6]) and Grosswald ((1) in [1]).

16. As our final result we shall utilize Theorem 6 to obtain a convergent series for $p(n)$, the number of unrestricted partitions of n. If

$$\bar{a} = \{1, 2, \cdots, (p-1)/2, p\}$$

then obviously $p_{\bar{a}}(n) = p(n)$. In this case we have

$$r = (p-1)/2, \qquad \bar{r} = p/2, \tag{16.1}$$

$$\overline{B} = \overline{A} = p/2 \text{ for all } b_1. \tag{16.2}$$

Since, therefore, $\overline{A}/12p = \overline{B}/12p = 1/24$ we see that (10.21) holds for all $n\geqq 1$ and that only $\nu=0$ appears in the first term. Furthermore, from (10.15) and (10.16) we have

$$\sum_{b_1=1}^{(p-1)/2} \overline{A}_{k,b}(n,0) = \overline{A}_k(n) = \sum_{h \bmod k}{}' \omega_{\bar{a}}(h,k)\exp\{-2\pi inh/k\}, \tag{16.3}$$

$$\overline{L}_{k,b}(n,0) = \frac{1}{k(24n-1)^{3/4}} I_{3/2}\{\pi(24n-1)^{1/2}/6k\}. \tag{16.4}$$

For all k such that $(p,k)=1$

$$\left(2^r\prod_{i=1}^{r}\sin\pi\alpha_i/p\right)^{-1} = \left(\prod_{j=1}^{(p-1)/2} 2\sin\pi j/p\right)^{-1} = p^{-1/2}. \tag{16.5}$$

Also, for all such k, the ρ_i in conjunction with the $\bar{\rho}_i$, run through the set of values $\omega, \omega^2, \cdots, \omega^{p-1}$, where $\omega = \exp\{2\pi i/p\}$, exactly once. Therefore, for all k

$$H_{\bar{a}}(x) = \prod_{n=1}^{\infty}\{(1-x^n)(1-\omega x^n)(1-\omega^2x^n)\cdots(1-\omega^{p-1}x^n)\}^{-1}.$$

Now $(1-z)(1-\omega z)\cdots(1-\omega^{p-1}z) = 1-z^p$ since both members are polynomials of the pth degree in z with roots $1, \omega, \cdots, \omega^{p-1}$. Therefore

$$H_{\bar{a}}(x) = \prod_{n=1}^{\infty} (1 - x^{np})^{-1} = 1 + c_{\bar{a}}^{(k)}(p)x^p + \cdots.$$

Since $\bar{r}/12 = p/24$ this shows that in the second term of (10.21) only $\nu = 0$ appears. From (10.19) and (10.20) we have

$$\overline{B}_k(n, 0) = \overline{B}_k(n) = \sum_{h \bmod k}{}' \chi_{\bar{a}}(h, k) \exp\{-2\pi inh/k\}, \tag{16.6}$$

$$\overline{L}_k^*(n, 0) = \frac{1}{k(24n - 1)^{3/4}} I_{3/2}\{\pi(24n - 1)^{1/2}/6k\}. \tag{16.7}$$

We conclude from (10.21) and (16.1)–(16.7) that

$$p(n) = \frac{2\pi}{(24n - 1)^{3/4}} \sum_{k=1}^{\infty} \frac{C_k(n)}{k} I_{3/2}\{\pi(24n - 1)^{1/2}/6k\} \tag{16.8}$$

where $C_k(n) = \overline{A}_k(n)$ if $p \mid k$ and $C_k(n) = \overline{B}_k(n)$ if $p \nmid k$.

From (16.3), (10.4), and the definitions of μ_i and μ when $p \mid k$ we see that

$$\overline{A}_k(n) = \sum_{h \bmod k}{}' \exp\left\{\pi i \sum_{j=1}^{k} ((j/k))((hj/k))\right\} \exp\{-2\pi inh/k\}.$$

Since $((j/k)) = j/k - [j/k] - 1/2 + (1/2)\delta(j/k)$ and since $\sum_{j=1}^{k} ((hj/k)) = 0$, we have

$$\overline{A}_k(n) = \sum_{h \bmod k}{}' \exp\left\{\pi i \sum_{j=1}^{k} \frac{j}{k} ((hj/k))\right\} \exp\{-2\pi inh/k\}. \tag{16.9}$$

Similarly,

$$\overline{B}_k(n) = \sum_{h \bmod k}{}' \exp\left\{\pi i \sum_{j=1}^{K} \frac{j}{pk} ((hj/k))\right\} \exp\{-2\pi inh/k\}.$$

Now

$$\begin{aligned} \sum_{j=1}^{K} \frac{j}{pk} ((hj/k)) &= \sum_{j=1}^{k} \frac{j}{pk} ((hj/k)) + \sum_{j=1}^{k} \frac{k+j}{pk} ((hj/k)) \\ &\quad + \sum_{j=1}^{k} \frac{2k+j}{pk} ((hj/k)) + \cdots + \sum_{j=1}^{k} \frac{(p-1)k+j}{pk} ((hj/k)) \\ &= p \sum_{j=1}^{k} \frac{j}{pk} ((hj/k)) = \sum_{j=1}^{k} \frac{j}{k} ((hj/k)). \end{aligned}$$

Therefore

$$\overline{B}_k(n) = \overline{A}_k(n). \tag{16.10}$$

From (16.8), (16.9), (16.10) we have

THEOREM 11. *The number, $p(n)$, of unrestricted partitions of a positive integer n is given by the convergent series*

$$p(n) = \frac{2\pi}{(24n-1)^{3/4}} \sum_{k=1}^{\infty} \frac{\overline{A}_k(n)}{k} I_{3/2}\{\pi(24n-1)^{1/2}/6k\} \tag{16.11}$$

where $\overline{A}_k(n)$ is given by (16.9).

This result is easily seen to be the same as that first proved by Rademacher ((1.8) in [7]).

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TEMPLE UNIVERSITY,
PHILADELPHIA, PENNSYLVANIA